Coherent Measures of Financial Risk

the Importance of Thinking Coherently

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Wednesday 9th November,
Kraków

SECTION 1

Introduction: What is Coherence and Risk?

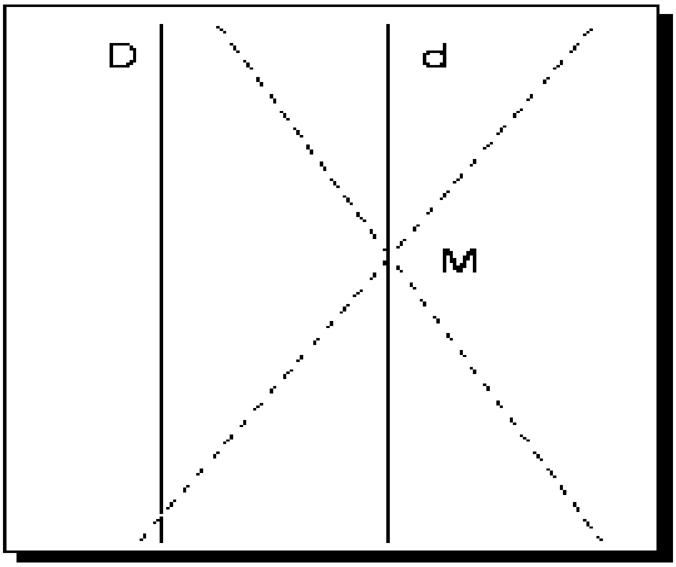

Figure 1 : Euclid Proposed 5 Axioms (or rather 3 + 2 definitions) in his "Elements" as foundation of Geometry. — see eg. (Heath 1909)

Alternative Coherent Geometries

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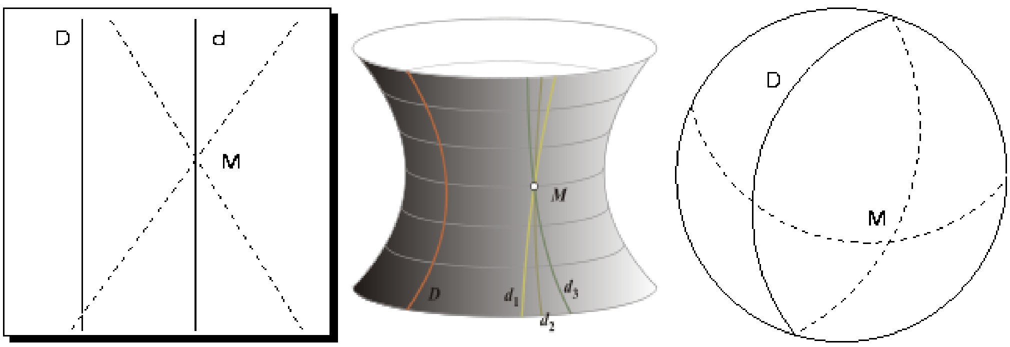
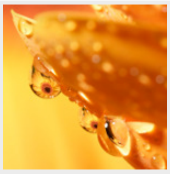


Figure 2 : Alternative coherent geometries. Where in Ecuclid's geometry there is exactly one line parallel to line D and through point M, in Nikola Lobatchevski's hypersphere there are an infinite number and in Bernhard Riemann's sphere there are none.



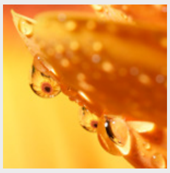
Thinking about Financial Risk

Idea	Reference
no risk, no rewards	Ecclesiastes 11:1–6 (ca. 300 BCE)
diversify investment	Ecclesiastes 11:1–2 (ca. 300 BCE) and Bernoulli (1738)

Table 1 : Key ideas about investment risk

Risk Measure	Reference
variance (**VAR**)	Fisher (1906), Marschak (1938) and Markowitz (1952)
Value at Risk (**VaR**)	Roy (1952)
semi-variance (**S**)	Markowitz (1991)
Expected Shortfall (**ES**)	Acerbi and Tasche (2002) and De Brouwer (2012)

Table 2 : Normative theories and their risk measures implied.



Definitions of Risk Measures I

Definition 1 (Standard Deviation / Variance)

$$VAR := \textbf{variance} = E[(X - E[X])^2]$$

$$\sigma := \text{standard deviation} = \sqrt{VAR}$$

Definition 2 (Value-at-Risk)

$$VaR_\alpha(\mathcal{P}) := -(\text{the best of the } 100\alpha\% \text{ worst outcomes of } \mathcal{P})$$

Definitions of Risk Measures II

Definition 3 (Expected Shortfall)

$$ES_{(\alpha)}(\mathcal{P}) := -(\text{average of the worst } 100\alpha\% \text{ realizations})$$

Definition 4 (Worst Expected Loss)

$$WEL := \textbf{Worst Expected Loss} = -E[\min(\mathcal{P})]$$

Visualization of some risk measures

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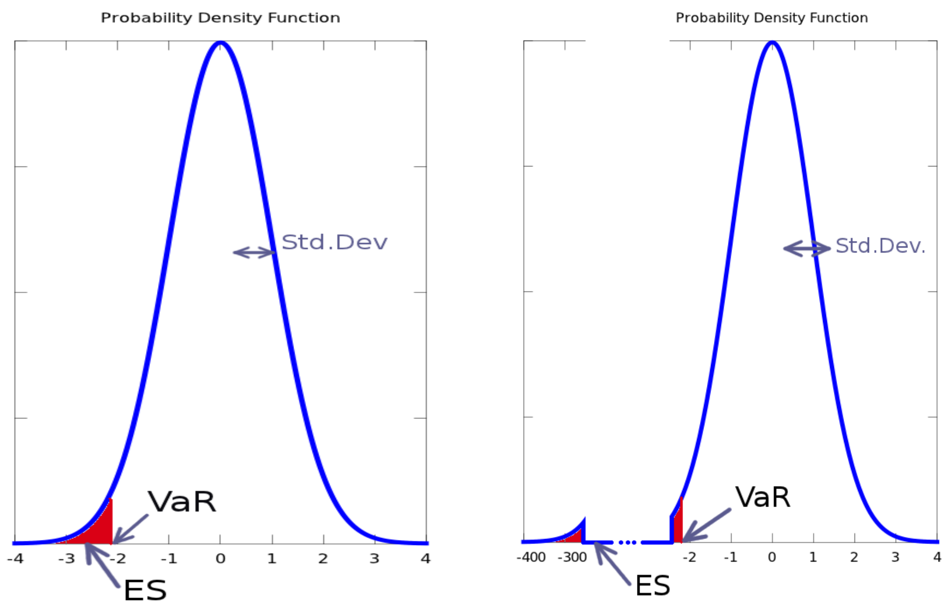
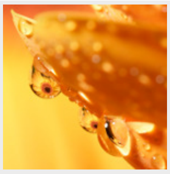
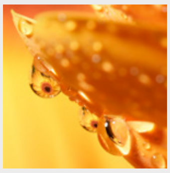


Figure 3 : visualization of ES, VaR and σ. Note that WEL is not defined.



AN AXIOMATIC APPROACH TO FINANCIAL RISK



A SET OF AXIOMS

PROPOSED BY ARTZNER, DELBAEN, EBER, AND HEATH (1997)

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DEFINITION 5 (COHERENT RISK MEASURE)

A function $\rho : \mathbb{V} \mapsto \mathbb{R}$ is called a **coherent risk measure** if and only if

1. **monotonous**: $\forall X, Y \in \mathbb{V} : X \leq Y \Rightarrow \rho(X) \geq \rho(Y)$
2. **sub-additive**: $\forall X, Y, X+Y \in \mathbb{V} : \rho(X+Y) \leq \rho(X) + \rho(Y)$
3. **positively homogeneous**: $\forall a > 0 \text{ and } \forall X, aX \in \mathbb{V} : \rho(aX) = a\rho(X)$
4. **translation invariant**: $\forall a > 0 \text{ and } \forall X \in \mathbb{V} : \rho(X+a) = \rho(X) - a$

Law-invariance under P:
$\forall X, Y \in \mathbb{V}$ *and* $\forall t \in \mathbb{R} : P[X \leq t] = P[Y \leq t] \Rightarrow \rho(X) = \rho(Y)$

Which Risk Measure is Coherent?

- **VAR (or volatility)** is not coherent because it is not monotonous (trivial)
- **VaR** is not coherent, because it is not sub-additive (Artzner, Delbaen, Eber, and Heath 1999)
- **ES** is coherent (Pflug 2000)
- **WEL** is not usable because it is not Law-Invariant

...but who should care?

Section 3

Case Studies

Case 1
One Bond

Example 1 (One Bond)

Assume one bond with a 0.7% probability to default in one year in all other cases it pays 105% in one year. What is the $1\% VaR$?

[$\mathcal{A}$] The 1% VaR is $-5\% \Rightarrow$ VaR spots **no** risk!

Case 2

Example 2 (Two Independent Bonds)

Consider two identical bonds with the same parameters, but independently distributed. What is the $1\% VaR$ now?

[$\mathcal{A}$] The 1% VaR of the diversified portfolio is 47.5%!

Case 4
The Evil Banker and his customers

Example 2 (The Evil Banker's First Dilemma)

Consider an Evil Banker who has to compose a portfolio for his private client. If there is at least one default in the portfolio, then the banker will loose that client.
How can our banker minimize his work and maximize his income?

[$\mathcal{A}$] The Evil Banker should minimize the probability that at least one bond defaults. This is:

$$P[\text{at least one default}] = 1 - \prod_{n=1}^{N} P[\text{one default}] = 1 - (0.7)^N.$$

The optimal value is hence $N = 1$.

Case 5
The Evil Banker and Basel III

Example 2 (The Evil Banker's Second Dilemma)

Consider an Evil Banker who has to comply with Basel III, hence uses for assessing market risk *VaR*. Being Evil he does not care about the size of a bailout. So how does he minimize VaR?

[$\mathcal{A}$] One bond is optimal. However, *VaR* only informs that there is 1% chance that the loss will be higher than the VaR. The Evil Banker does not care, but the society should care about the size of an eventual bailout.

Case 6
More Bonds

Example 3 (N Independent Bonds)

Consider now an increasing number of independent bonds with the same parameters as in previous example.
Trace the risk surface.

Risk in Function of Diversification
Convecity (I)

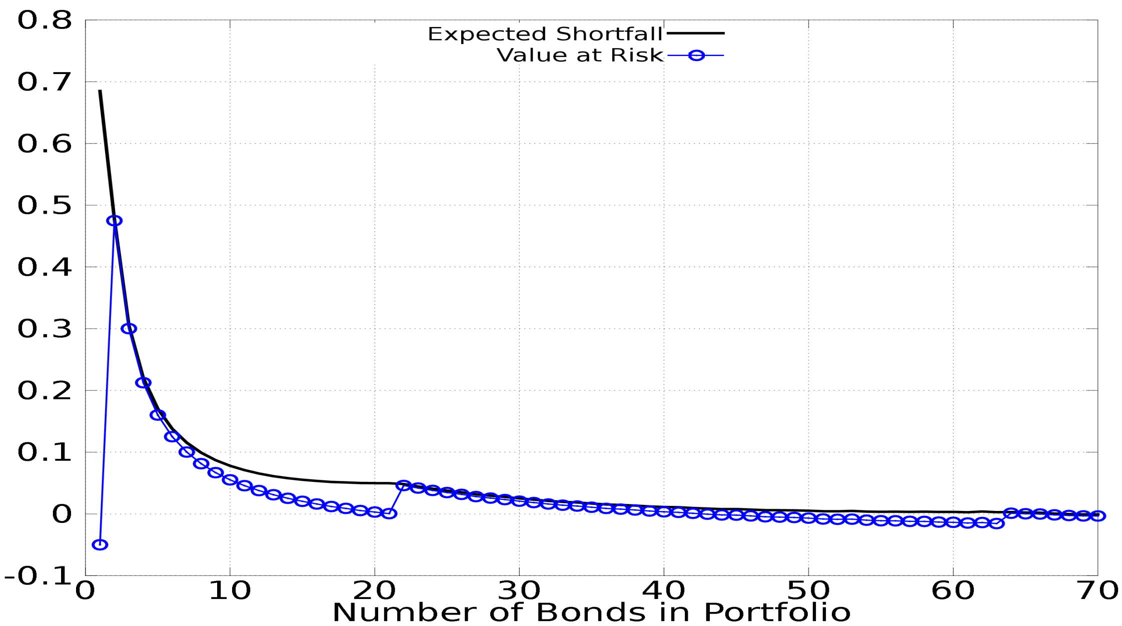

Figure 4 : ES and VaR in function of number of bonds.

Incoherent risk measures in legislation

legislation	"risk measure"	result
UCITS	VaR and VAR	non suitable assets
Basel	VaR	crisis
Solvency	VaR	insolvency

Table 3 : Law makers increasingly use non-coherent risk measures in legislation, resulting in encouraging to take large bets, ignore extreme risks and mislead investors. All building up to the next crisis …building up to the next global disaster.

Yes, it is important!

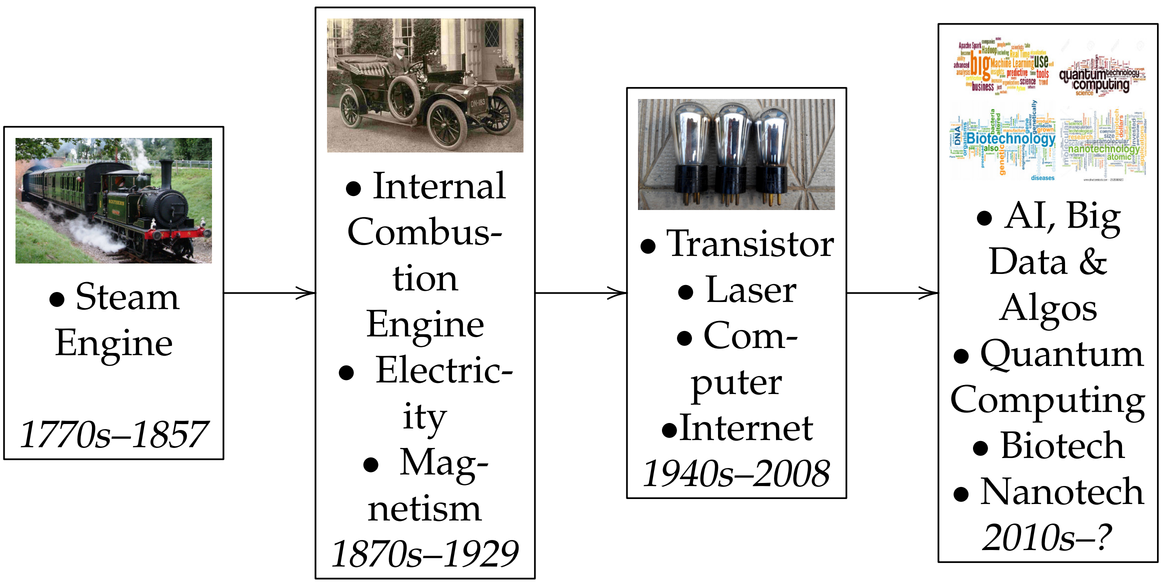

Figure 5 : A simplified model of science propelling welfare and economy, but leading to crisis situations.

THE LIMITS OF COHERENT RISK MEASURES

THE LIMITS OF COHERENT RISK MEASURES
LIQUIDITY

EXAMPLE 4 (ILLIQUID ASSETS)

Imagine that you hold twice the average daily volume in stock X. Is it realistic to demand from a risk measure that it is positive homogeneous and hence that $\forall a > 0$ and $\forall X, aX \in \mathbb{V} : \rho(aX) = a\rho(X)$?

EXAMPLE 5 (THIRSTY)

Imagine that you need to drink in order to cross the desert, but you know that one of your five bottles is poisoned (of course you don't know which one). What strategy do you take to minimize risk? Diversify or Russian Roulette?

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EXAMPLE 6 (BASEL II WITH ES?)

Would it make sense to replace *VaR* in the capital requirements for banks by *ES*?

[$\mathcal{A}$] It would be a significant improvement, but would it also not work systemic? (i.e. act as a non-linear feedback system in case of disaster)

EXAMPLE 7 (RISK AND REWARD INDICATOR?)

Could a coherent risk measure be a "risk and reward indicator"?

[$\mathcal{A}$] Stochastic Dominance of Second Order implies dominance of *ES* (Yamai and Yoshiba 2002). However for *ES* to imply stochastic dominance of the second order–and hence imply preference in utility theory–one would need an infinite number of *ES* calculations for all $\alpha \in [0, 1]$.

CONCLUSIONS

Conclusions

Coherence does matter and its importance cannot be underestimated

1. Coherence does matter.
2. An incoherent risk measure will lead to counter-intuitive and dangerous results.
3. Hence, it is worth to make a rough estimate about the left tail of the distribution rather than ignoring it.
4. Also Coherent Risk measures are a simplified reduction of the complex reality

Back-matter

BIBLIOGRAPHY

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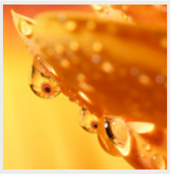
Section 8

Nomenclature



NOMENCLATURE I

ρ	a risk measure, $\rho : \mathbb{V} \mapsto \mathbb{R}$, page 12–17
$ES_\alpha(\mathcal{P})$	Expected Shortfall = the average of the $\alpha 100\%$ worst outcomes of $\mathcal{P}$; aka CVaR, Tail-VaR, etc., page 8
$VAR(X)$	Variance: $VAR(X) = E[X^2] - E[X]^2 = \sigma^2$, page 7
$VaR_\alpha(\mathcal{P})$	Value at Risk, page 7
BCE	Before Common Era, page 5, 6
WEL	Worst Expected Loss, page 8