The fallacy of large numbers revisited: The construction of a utility function that leads to the acceptance of two games, while one is rejected

Received: 15th June, 2000

Philippe De Brouwer*
holds degrees in Physics and Economics, and is currently Director at Fortis Investment Management. Philippe De Brouwer graduated in theoretical physics (University of Brussels) and published some work in the field of non-linear dynamics and chaos theory. Afterwards, he oriented his career towards financial economics and specialised in commercial engineering (University of Brussels). Now after the recent merger with Fimagen, and the creation of FIM (Fortis Investment Management), he is Director and is responsible for strategy and co-ordination.

Freddy Van den Spiegel
holds a PhD in Economics, and is currently Chief Economist of Fortis Bank, Professor at the Free University of Brussels, and Member of the Executive Committee of Association Belge des OPC. He has an impressive track record in asset management and was the founder of Fortis Investments Belgium. Professor Van den Spiegel is an active academician and a speaker at many conferences.

*Fortis Investment Management, Boulevard Roi Albert II, 1, 1210 Brussels, Belgium.
Tel: +32 2274 8487; Fax: +32 2274 8206; e-mail: philippe.debrouwer@fimgroup.com

Abstract Should the composition of an investment portfolio differ depending on whether the investment horizon is longer or shorter? Practitioners have always allocated more risky investments to portfolios with longer investment horizons. But in 1963, P. A. Samuelson put forward strong arguments that this is an erroneous approach. Since then, the financial community has lived with this conflict and searched for plausible explanations. In this paper, however, we construct a counterexample to the thesis of 1963.

Keywords: *law of large numbers; investment horizon; portfolio construction; decisions under uncertainty; asset allocation*

Introduction

One of the most intriguing debates of recent years is without any doubt the question of whether the composition of an investment portfolio should vary with the investment horizon. According to the theory of Markowitz (1952), the answer seemed to be 'yes'; however, this was only until Samuelson added two other important papers to his already impressive list of publications: one tackling the generally accepted 'law of large numbers' (1963), and a second (1969) showing that the composition of an optimal portfolio under certain conditions is constant during the lifetime. The conditions necessary for this result are thoroughly studied, for example by Samuelson himself (1994), or Fisher and Statman (1999). In our opinion, these conditions do not seem to be the generic case, and therefore their importance is rather academic.

The thesis of the first article (Samuelson, 1963), however, remained unchanged: 'a person whose utility schedule prevents him from ever taking a specific favourable bet when offered only once can never rationally take a large sequence of such fair bets, if expected utility is maximised'. This is of utmost importance, since it is a major argument in this discussion. If 'the law of large numbers' is a fallacy, as published by Samuelson in 1963, then there is no additional security in large numbers. Therefore, longer horizons (adding large sequences or returns) will not add any security either, and the composition of a financial portfolio should not depend on the investment horizon of the holder.

In this paper, we briefly outline the discussion from Samuelson's (1963) paper. We then construct a utility function, which shows that when challenging a person to the very same game proposed by Samuelson, the person will not accept one game but will accept two or more. Further, we explore the relationship with real investments, and finally we formulate some guidelines for financial advisors.

The game

The idea is as follows: 'Would you accept a bet on tossing a coin? You get \$200 if your guess is right, you pay \$100 if you lose.' In his famous article, Samuelson (1963) describes how he proposed this bet to a colleague,[1] who declined, since the loss of \$100 would harm him more than \$200 could bring benefit. The friend adds that he would accept a large series of such games: 'I'll take you on if you promise to let me make 100 such bets.'

Samuelson then proves that 'unfairness can only breed unfairness' and therefore 100 bets cannot be a favourable gamble if one is not'. Later, he states that 'he [SC] should have asked to subdivide the risk and asked for a sequence of 100 bets, each of which was 100th as big (or \$1 against \$2)'. This should make us suspicious, considering that 100 bets of \$1 against \$2 are acceptable, and one big game is not acceptable. If this is true, then it should also be possible that the acceptance of 100 big bets differs from one.

The very essence of Samuelson's reasoning is based on the fact that one can subdivide a series of games into constituent games. Therefore, he argues, even if we would accept a first game (knowing that we get a second), then we will decline the last (since this is one game and we do not want one game), therefore the first game needs to be rejected too. This is not true. The outcome of two games does have a different probability density function, and therefore it can lead to different expected utility.

In the following section, we present a utility function that leads to the rejection of one game, but also to the acceptance of two games.

A counterexample

Let us now consider a simple utility function which breaks this theorem. Assume that our Bernoulli[2] utility function is given by

$$U(W) = \begin{cases} qW & \text{if } W < W_0 \\ W + (q-1)W_0 & \text{if } W \geq W_0 \end{cases}$$

where W stands for the 'wealth', W_0 is a threshold, and q is a positive constant larger than one.

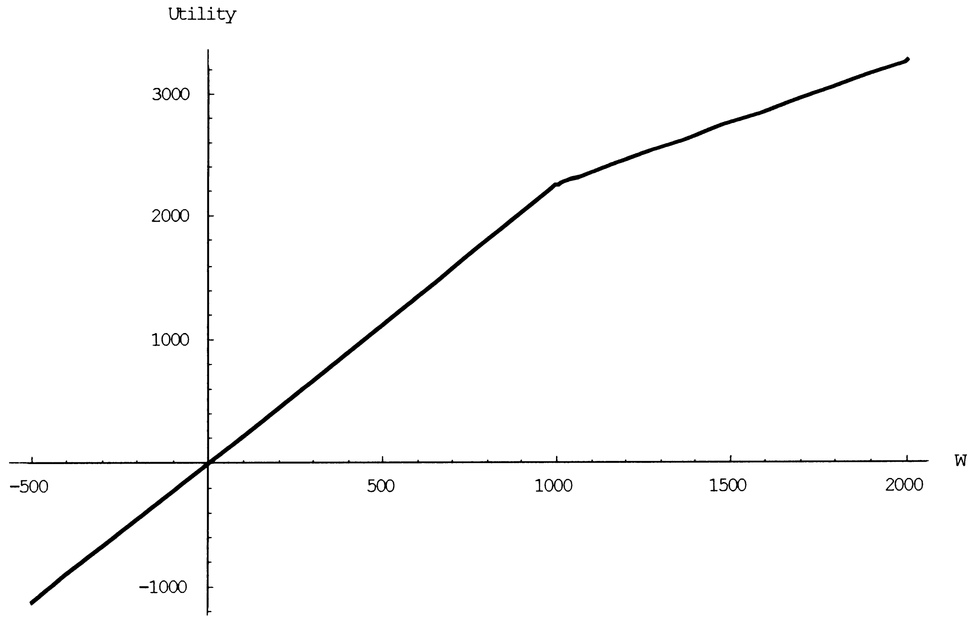

Figure 1 The Bernoulli utility function. Wealth is shown on the horizontal axis and the vertical axis is a scale for utility

The graph of this utility function is shown in Figure 1. This kind of utility function shows that the investor has an important threshold (W_0). This can be considered as a consequence of a certain 'aspiration level' (described in the SP/A theory; see Lopez, 1987), or it can be considered as the effect of the loss aversion of the investor. This closely relates to the prospect theory as formulated by Kahneman and Tversky (1979).[3] Please note that there is no contradiction between these two formulations.

This corresponds, of course, to the fact that Samuelson's colleague said that the loss of \$100 would hurt more than the benefit of a gain of \$200.

We understand that this utility function does allow for 'a Saint Petersburg paradox'. This means that it can lead in some extreme cases to diverging results. Indeed, one can construct a game with infinite expected utility, but of such a nature that no person will pay such an amount to participate in that game. This argument has its theoretical importance; however, this function can be considered as an approximation in the vicinity of W_0. For very large values of W, it might be considered as bounded.[4]

Like Benartzi and Thaler (1995), we choose q equal to 2.25. The fact that q is bigger than unity reflects 'loss aversion': people tend to value losses much more than profits of equal size. Besides, q has to be bigger than 2 if we want to use a utility function that explains SC's remark about the fact that a loss of \$100 weights more than a profit of \$200. Using $\log(W)$, for example, ignores SC's important remark.

The value of W_0 has been chosen to equal \$1,000. We thus obtain the following results for the von Neumann utility (= 'expected utility') for no game, one and two games (as described in the previous section, win is \$200, loss is \$100). We denote $U_{\mathrm{v.N-M}}(f_n)$ as the

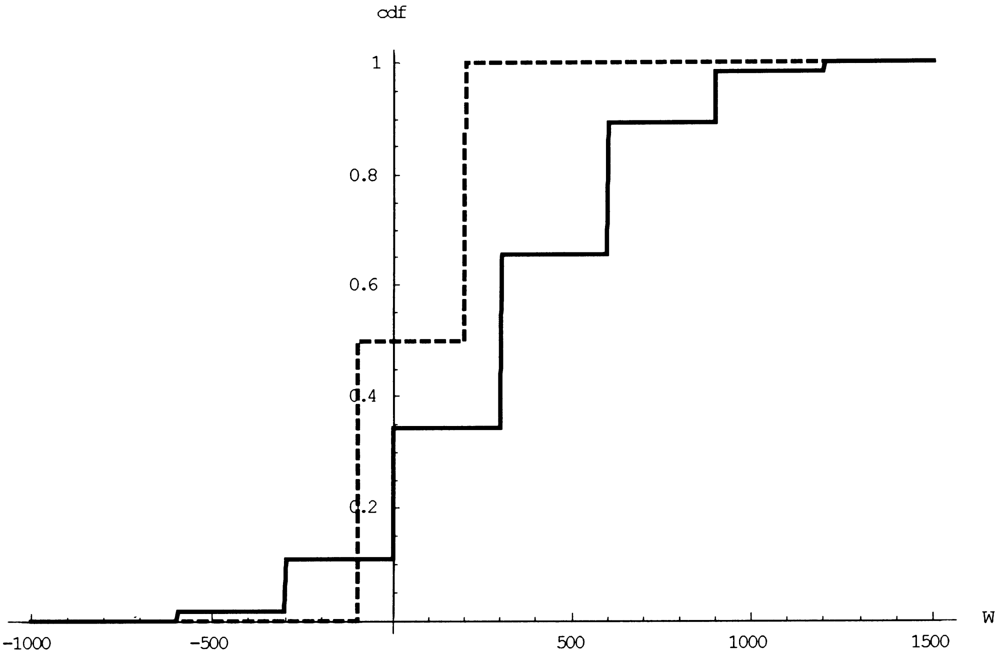

Figure 2 Two cumulative distribution functions (cdf) for Samuelson's game. The dotted line is one bet, and the solid line is the cdf for the game with six tosses

v.N–M utility for *n* plays.

$$U_{\text{v.N–M}}(f_0) = 2{,}250.0$$
$$U_{\text{v.N–M}}(f_1) = 50\%\ (2{,}025) + 50\%(2{,}450) = 2{,}237.5$$
$$U_{\text{v.N–M}}(f_2) = 25\%(1{,}800) + 50\%(2{,}350) + 25\%(2{,}650) = 2{,}287.5$$

It can be seen that the v.N–M utility for one game is lower than the expected utility for no game. Therefore, a person who acts according to the framework of maximising utility will not accept one game.

But, the v.N–M utility for two games is higher than for one game and even higher than the utility of not playing at all. This proves that *a person whose utility schedule prevents him from taking a specific favourable bet when offered only once can rationally take a large sequence of such fair bets, if expected utility is maximised.*

The following section is devoted to financial assets. One can observe the same shift in statistical distributions of financial assets as in the game, *therefore it is possible that risky investments are not acceptable to somebody's utility schedule on a short-term horizon, but are well suited to investments on longer horizons.*

Long-term investments

In the previous section, we proved that it is possible within the framework of utility theory to turn down one bet, but accept a series of them. The remaining challenge is to explore the relevance for decisions about financial investments. First, let us consider the cumulative distribution functions (cdf) for Samuelson's game (see Figure 2).

In order to calculate the corresponding cdfs for the investment problem we have to make some assumptions about the underlying distributions. It is accepted now that financial assets do not follow a Gaussian distribution (see, for example,

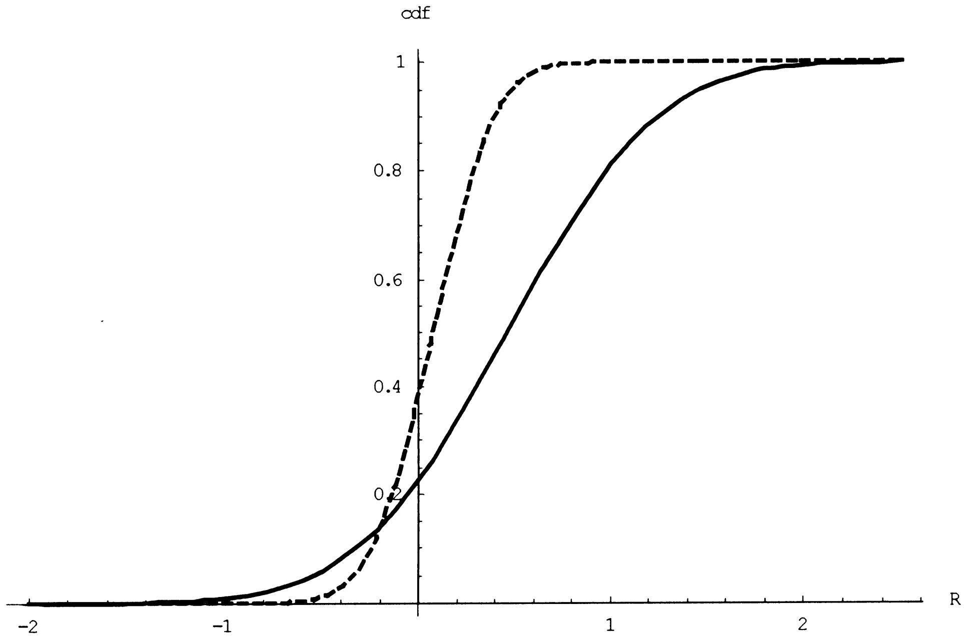

Figure 3 The cdf for the return (R) of a financial investment (under the assumptions described above) for a horizon of 1 year (dotted line) and 6 years (solid line)

Bouchard and Potters (1997), and many more authors). However, we are interested only in qualitative analyses, so we can use the Gaussian distribution as a good approximation. Besides, the utility function itself can be only a reasonable approximation. The exact numeric result is not important, but the stability for different utility and distributions functions counts.

Furthermore, we shall work with the logreturns for reasons of additivity and define them as follows.

$$R_t = \log\left(\frac{S_t}{S_{t-1}}\right)$$

where S_t is the price of the asset at time t. This assumption allows us to draw the cdf of financial assets.[5] In Figure 3 we show the cdf of the logreturns on one year (dotted line), and 6 years (solid line). The similarity with the games as described earlier is striking.

Longer horizons display more diffusion, but the mean moves to the right. We also notice that the longer one holds a stock, the more can be lost, but the probability of having a higher return increases quickly.

All this is very similar to the game that Samuelson described in 1963. Indeed, using the same assumptions and parameters as mentioned before, we can calculate the expected utility of an investment with a horizon of 0 (this is no investment), 1 and 2 years.

$$U_{\text{v.N–M}}(f_0) = 2{,}250.00$$
$$U_{\text{v.N–M}}(f_1) = 2{,}244.53$$
$$U_{\text{v.N–M}}(f_2) = 2{,}307.36$$

(f_n is now the cdf of a financial investment over n years.)

Again we notice that playing one game decreases our initial utility, but a set of two plays increases our utility. In this case, however, the result should be read as 'investing for one year in stocks is unacceptable for this individual, but on a two-year horizon, it becomes acceptable'.

Conclusion

The results obtained in this paper indicate that an individual who seeks to increase his utility might refuse to select some risky assets if the investment horizon is too short. These investments might be well suited for the same individual, if the investment horizon is long enough. This result indicates that there is a relationship between the composition of the best-suited investment portfolio and the investment horizon.

Postscript

This result is reassuring for financial advisors, who allow their advice to be dependent on the investment horizon. Now that the question of whether an investment portfolio should be dependent on the horizon is answered positively, another even weightier question arises: 'How can I find out which portfolio is suited for which investment horizon for this investor?'

We do not pretend to have a complete answer to this difficult question, but hope to be of some help with the following suggestions. The following list is a first attempt to bridge the gap between theory and practice; it is not exhaustive and some suggestions even overlap to some extent. We believe, however, that it is already a good checklist.

Suggestion 0: Have a decision process

Find out how a portfolio can be constructed; try to cover first the major problem, namely the strategic asset allocation. Brinson *et al.* (1986, 1991) show that 90 per cent of the variability of a fund is due to the asset allocation. Tactical asset allocation and stock picking are therefore much less important.

Suggestion 1: Try to understand the investment problem

It is essential to understand that investment advice can only be relative to an investment problem, in other words the goal(s) of the investor. There is no such thing as 'the stock of the week'; there is no good solution for all investment problems.

The goal of the investor is an important issue, since it determines the investment horizon.

Two kinds of investment problems will arise: integrated and isolated goals. Investors tend to divide their portfolio into 'mental accounts';[6] each sub-portfolio is at least mentally isolated and has its own goal. This situation is very different from an integrated portfolio covering all future needs and wishes.

Suggestion 2: Give your client some insight into the problem

Talks about economic scenarios or forecasts of a particular company are nice and might be fascinating to your client, but they are only related to the last step in our decision process. Try to focus on insight: what is the essential difference between a bond, a stock and a mixed portfolio? Why should we first focus on a strategic asset allocation and determine a benchmark?

This approach has more added value for the investor, and aims more at a long-term relationship.

Suggestion 3: Provide relevant information

If (from suggestion 1) you learned that your client has a long investment

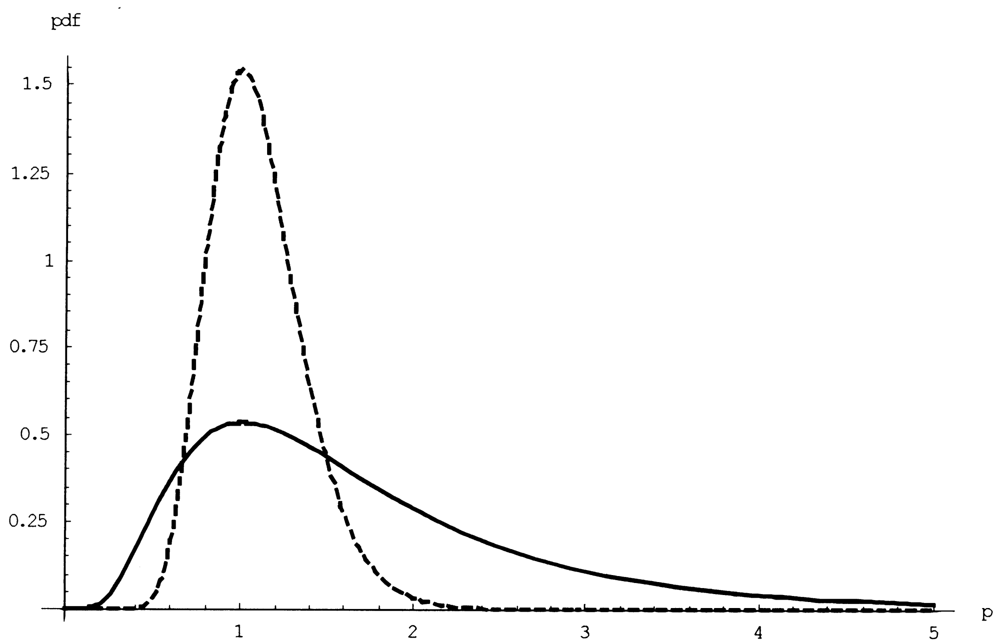

Figure 4 The probability density function of the price (*p*) of the asset; the initial price is unity on the horizontal axis, the dotted line is 1 year, the solid line 6 years

horizon (eg pension plan savings), it is of limited relevance to show short-term returns or forecasts. Short-term returns can have some influence on the decisions of the investor; however, the return over his personal horizon is the essential information.

Consider also whether it is important to explain about returns or about the changes in value of the portfolio, as ultimately it is the value of the portfolio that determines the purchasing power. It is a very difficult task, even for specialists, to make an estimation of the cumulative effect of a series of returns. In other words: select the widest 'frame' possible.[7]

Are historic data relevant to provide information about the future? Historical data are certainly objective information, but be aware of 'Peso effects'. Some fundamental changes or paradigm shifts might occur in the economy.

Suggestion 4: Visualise the impact of a decision and keep it simple

Decisions about financial investments are of uppermost importance in our life, but few people manage to get a good view of the financial products and investment markets. Describing the alternatives will help, but a visual aid will provide a deeper understanding.

In a recent publication by Benartzi and Thaler (1999), we find a very interesting example of using such visual aids in financial decision making. When long-term returns are shown to a panel (rather than short-term returns), people are more likely to invest more of their retirement savings in stocks. Another finding in the same paper is that higher education minimises the selection of irrational combinations. Among other possible explanations, we feel that the understanding of the problem influences the result. Higher education (MBA in

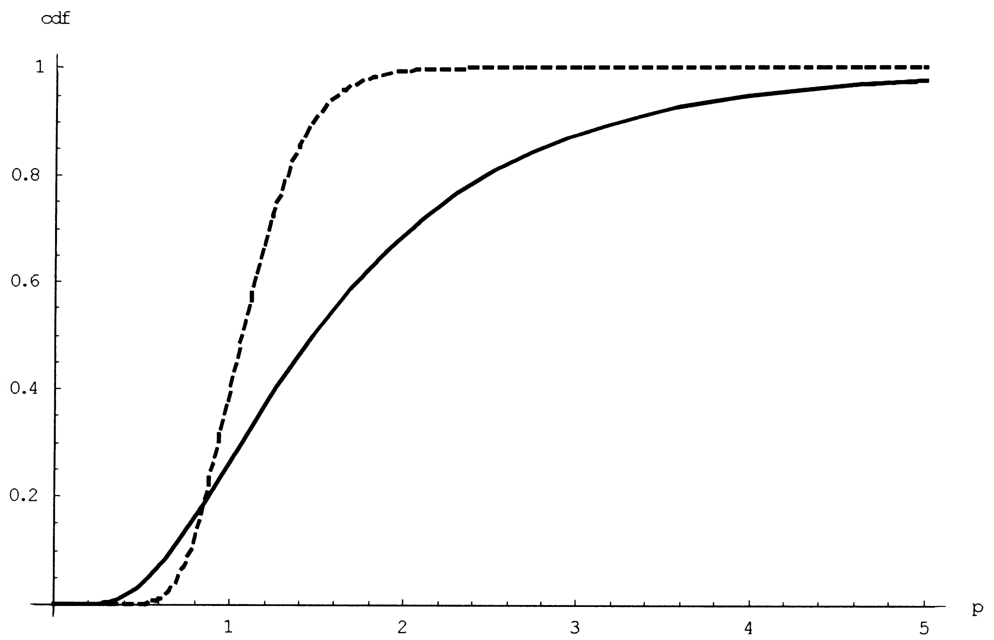

Figure 5 The cumulative distribution function of the price (*p*) of the asset; the initial price is unity; the dotted line is 1 year, the solid line 6 years

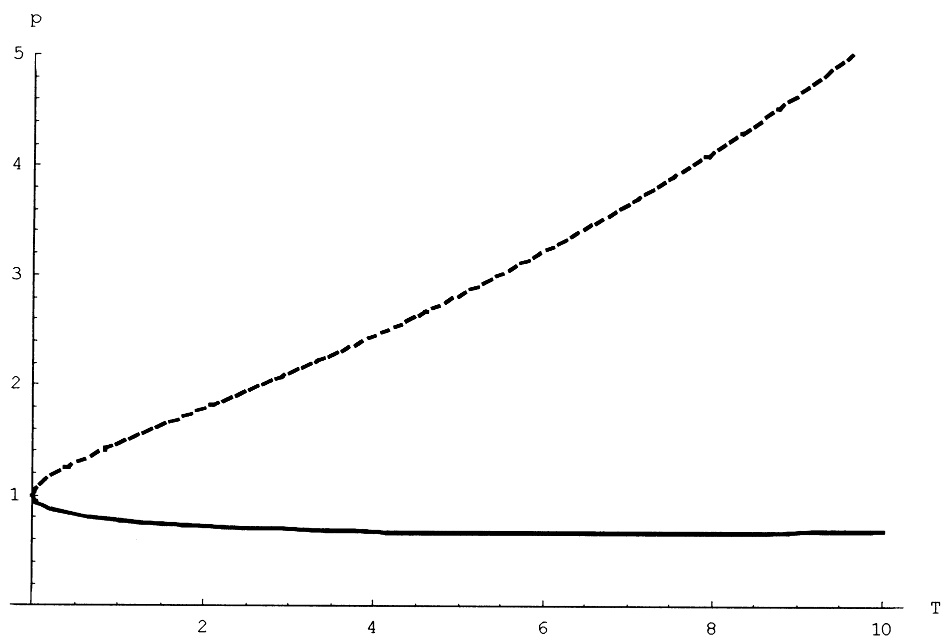

Figure 6 A plot of the evolution of 10 per cent (solid line) and 90 per cent (dotted line) quantiles, now you find on the horizontal axis the time (*T*) and on the vertical axis the value of the asset (*p*)

their study) guarantees some exposure to statistics that will help to understand the relation between one bet and repeated bets (or the relations between sequences of returns and purchasing power).

Human beings tend to use narrow frames when making decisions; therefore, the financial advisor should try to use the widest frame possible. Wealth is a wider frame than return, but it is as difficult to find a way to visualise it as it is for 'return'.

We leave it to the reader to consider Figures 4–6 carefully, comparing their information content and degree of complexity. (Think also about suggestion 3.) These graphs might as well be constructed from the histogram of historical observations.

Suggestion 5: Avoid presentation bias

Selecting financial investments seems to rely on some fundamental heuristics, deeply rooted in our genes. The $1/n$ rule is such a secure refuge.

Again we refer to the work of Benartzi and Thaler (1998). They study the relation between the proportion allocated to equities (versus bonds) and the nature and number of the funds that are presented in pension schemes. Employees of companies that are in all aspects similar allocate different parts of their portfolios to equities. If the scheme presents k equity funds and l bond funds, the portion allocated to equities tends to $k/(k+1)$, no matter what the value of k or l is!

This shocking result shows that when the employee has to pick funds (the 'USA system') it is actually the employer who determines the asset allocation on average, only by selecting the funds in the scheme. An alternative is to allow the individual to choose between different portfolios (different compositions) on the basis of historical returns (as used in Chile). However, it might be that lots of people will just pick the centre portfolio.

Financial investors bear an important responsibility no matter what techniques are used!

Notes

1 Like Tversky and Bar-Hillel (1983), we refer to Samuelson's colleague as 'SC'.

2 We follow the terminology introduced by Mas-Collel *et al.* (1995). We do so since we consider it very important to understand the difference between the utility in function of a given level of wealth, and the utility of a lottery or investment, where the final result on the wealth is yet unknown. The Bernoulli (B) utility function describes the utility when the outcome of the investment is known, and the von Neumann–Morgenstern (v.N–M) utility is the expected utility of the previous (thus the utility on which we base our decisions). If the axioms of von Neumann and Morgenstern hold, then we have:

$$U_{\text{v.N–M}}(f_X) = \int_{-\infty}^{+\infty} U_B(x) f_X(x)\,\mathrm{d}x$$

where f is the probability density function related to the investment or lottery being considered, whose return is the stochastic variable X.

3 The utility function of Kahneman and Tversky has a strict positive second derivative for losses and a negative one for gains; in other words, the investor is risk-averse when gains are involved and risk-seeking when losses are involved. Here, we use a simpler utility function, since we are just constructing an example. More complex utility functions that are in agreement with this theory can be constructed, for example,

$$U_B(W) = \begin{cases} q\,\log(W_0 + 1 - W) + U_0 & \text{if } W < W_0 \\ \log(W + 1 - W_0) + U_0 & \text{if } W \geq W_0 \end{cases}$$

For this particular form and parameters q and W_0 equal to those defined above, one will notice that one or two games is not acceptable, but from three games on the expected utility is above the utility of no game.

4 In the context of the Saint Petersburg paradox, it is important to understand that for every non-bounded utility function, one can construct a game that leads to infinite expected utility, but for which no human is willing to pay infinity. Menger (1934) first mentioned this theorem.

5 This and all further graphs are based on a Gaussian distribution with a mean return of 8 per cent annually and a volatility of 25 per cent.

6 This is a very common situation for practitioners, but it only recently attracted some academic interest. This phenomenon is described in the 'behavioural

Portfolio Theory' (see, for example, Shefrin and Statman, 1999).

7 The psychological bias to omit some factors while making decisions (eg ignore correlations) was called 'narrow framing' by Kahneman and Lovallo (1993).

References

Benartzi, S. and Thaler, R. H. (1995) 'Myopic Loss Aversion and the Equity Premium Puzzle', *Quarterly Journal of Economics*, February, 73–92.

Benartzi, S. and Thaler, R. H. (1998) 'Naïve Diversification Strategies in Defined Contribution Saving Plans', forthcoming in the *American Economic Review*.

Benartzi, S. and Thaler, R. H. (1999) 'Risk Aversion or Myopia? Choices in Repeated Gambles and Retirement Investments', *Management Science*, 45, March, 364–381.

Bouchard, J.-P. and Potters, M. (1997) *Théorie des risques financiers*, Commissariat à l'Energie Atomique, Paris (English edition published 1999 by Cambridge University Press).

Brinson, G. R., Randolph Hood, L. and Beebower, G. L. (1986) 'Determinants of Portfolio Performance', *Financial Analysts Journal*, July–August, 39–44.

Brinson, G., R., Singer, D. and Beebower, G. L. (1991) 'Determinants of Portfolio Performance II: An Update', *Financial Analysts Journal*, May–June, 88–97.

Fisher, K. L. and Statman, M. (1999) 'A Behavioural Framework for the Time Diversification', *Financial Analysts Journal*, May–June, 00–00.

Kahneman, D. and Lovallo, D. (1993) 'Timid Choices and Bold Forecasts: A Cognitive Perspective on Risk Taking', *Management Science*, 39, 17–31.

Kahneman, D. and Tversky, A. (1979) 'Prospect Theory: An Analysis of Decision under Risk', *Econometrica*, 4, 263–91.

Lopez, L. (1987) 'Between Hope and Fear: The Psychology of Risk', *Advances in Experimental Social Psychology*, 21, 255–95.

Markowitz, H. M. (1952) 'Portfolio Selection', *Journal of Finance*, 6, 77–91.

Mas-Colell, A., Whinston, M. D. and Green, J. R. (1995) *Laws of Economics: Microeconomy*, Oxford University Press, Oxford.

Menger, K. (1934) 'Das Unsicherheidsmoment in de Weltlehre', *Zeitschrift für Nationalökonomie*, 51.

Samuelson, P. A. (1963) 'Risk and Uncertainty: A Fallacy of Large Numbers', *Scientia*, April–May.

Samuelson, P. A. (1994) 'Lifetime Portfolio Selection by Dynamic Stochastic Programming', *Review of Economics and Statistics*, August.

Samuelson, P. A. (1989) 'The Long Term Case for Equities', *Journal of Portfolio Management*, Fall, 15–24.

Shefrin, H. and Statman, M. (1999) 'Behavioural Portfolio Theory', Working Paper, Santa Clara University.

Tversky, A. and Bar-Hillel, M. (1983) 'Risk: The Long and Short', *Journal of Experimental Psychology: Human Learning, Memory and Cognition*, 9, 713–17.

