Coherent Measures of Financial Risk
On the Importance of Thinking Coherently

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Coherent Risk Measures

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INTRODUCTION: WHAT IS RISK?

THINKING ABOUT FINANCIAL RISK

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- ca. 300 BCE
  - No Risks, No Rewards (Ecclesiastes 11:1-6)
  - Diversify your investments (Ecclesiastes 11:1-2)
- diversification reduces risk (Bernoulli 1738)
- variance could be a measure for economic risk (Fisher 1906)
- use mean and variance in utility (Marschak 1938)
- mean-variance criterion (Markowitz 1952a)
- semi-variance is better (Markowitz 1959)
- semi-variance relative to investment goal is "more plausible than variance" (Markowitz 1991)

$$S := E[\min(0, R - c)^2]$$

Defining Financial Risk

- Markowitz (1952a): variance is ok, because there is no important utility function that is compliant with semi-variance but not compliant with variance.
- HOWEVER, risk is relative to investment goal (see Markowitz (1952b) and De Brouwer (2009)) $\Rightarrow$ utility is compliant with S and not with variance (VAR)

Some Definitions I

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Definition 1

$$\begin{aligned}
\mathbb{V} &:= \text{the set of the real valued stochastic variables} \\
X &:= \text{a stochastic variable, with } x \text{ a realization} \\
E[X] &:= \text{the expected value of a stochastic variable } X \\
f_X &:= \text{its probability density function (pdf)} \\
F_X &:= \text{its cumulative distribution function} \\
f^{-1}(.) &:= \text{the inverse of function f} \\
\alpha &:= \text{a probability } \in [0,1] \\
\mathcal{P} &:= \text{the absolute return ``profit'' } (\mathcal{P} \in \mathbb{V})
\end{aligned}$$

Some Definitions II

Note that $\mathcal{P} = -\mathcal{L}$ (the "loss", expressed in monetary terms).

Definition 2 (Standard Deviation / Variance)

$$VAR := \textbf{variance} = E[(X - E[X])^2]$$
$$\sigma := \text{standard deviation} = \sqrt{VAR}$$

Definition 3 (Quantile Function)

$$Q_X(\alpha) := F_X^{-1}(\alpha) = inf\{x \in \mathbb{R} : \alpha \leq F_X(x)\}$$

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Some Definitions III

Definition 4 (Value-at-Risk)

$$\begin{aligned} VaR_\alpha(\mathcal{P}) &:= -Q_{\mathcal{P}}(\alpha) \\ &= -(\text{the best of the } \alpha 100\% \text{ worst outcomes of } \mathcal{P}) \end{aligned}$$

Definition 5 (Worst Expected Loss)

$$WEL := \textbf{Worst Expected Loss} = -E[\min(\mathcal{P})]$$

Some Definitions IV

Definition 6 (Expected Shortfall)

$$\begin{aligned} ES_{(\alpha)}(\mathcal{P}) &= -\frac{1}{\alpha}\int_0^{\alpha} Q(p)\,\mathrm{d}p \\ &= -\frac{1}{\alpha}\int_0^{\alpha} VaR_{(\alpha)}(\mathcal{P})(\mathfrak{p})\,\mathrm{d}\mathfrak{p} \\ &= -\frac{1}{\alpha}\int_{-\infty}^{Q_{\mathcal{P}}(\alpha)} f_{\mathcal{P}}(\mathfrak{p})\,\mathrm{d}\mathfrak{p} \\ &= -(\text{average of the worst } 100\alpha\% \text{ realizations}) \end{aligned}$$

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Some Definitions V

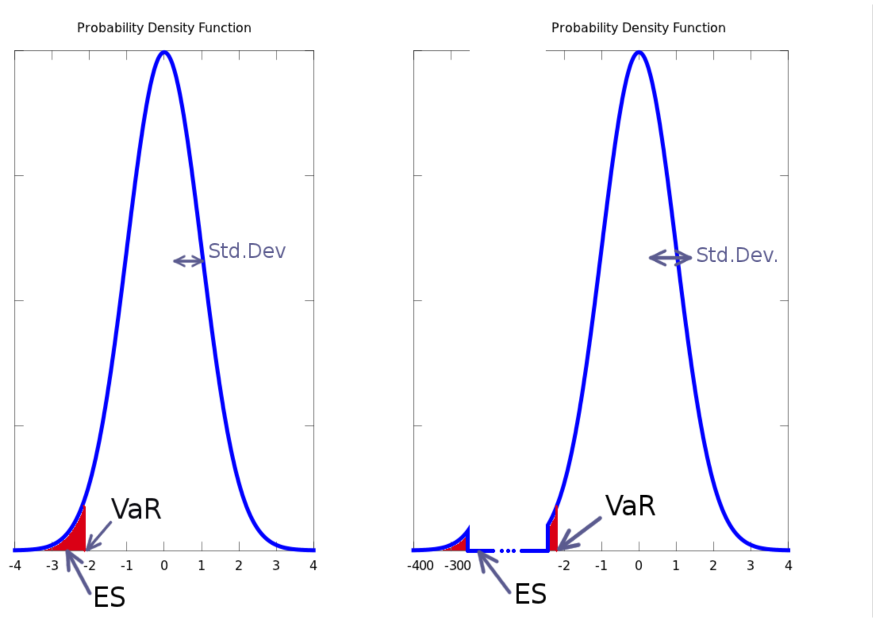

Figure 1: Interpretation of ES, VaR and σ.

Section 2

An Axiomatic Approach to Financial Risk

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A set of Axioms

Proposed by Artzner, Delbaen, Eber, and Heath (1997)

Definition 7 (Coherent Risk Measure)

A function $\rho : \mathbb{V} \mapsto \mathbb{R}$ is called a **coherent risk measure** if and only if

1. **monotonous**: $\forall X, Y \in \mathbb{V} : X \leq Y \Rightarrow \rho(X) \geq \rho(Y)$
2. **sub-additive**: $\forall X, Y, X + Y \in \mathbb{V} : \rho(X + Y) \leq \rho(X) + \rho(Y)$
3. **positively homogeneous**: $\forall a > 0$ and $\forall X, aX \in \mathbb{V} : \rho(aX) = a\rho(X)$
4. **translation invariant**: $\forall a > 0$ and $\forall X \in \mathbb{V} : \rho(X + a) = \rho(X) - a$

Law-invariance under P:
$\forall X, Y \in \mathbb{V}$ *and* $\forall t \in \mathbb{R} : P[X \leq t] = P[Y \leq t] \Rightarrow \rho(X) = \rho(Y)$

Which Risk Measure is Coherent?

- VAR (or volatility) is not coherent because it is not monotonous (trivial)
- VaR is not coherent, because it is not sub-additive (Artzner, Delbaen, Eber, and Heath 1999)
- ES is coherent (Pflug 2000)

...but who should care?

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Section 3

Case Studies

Case 1
Two Bonds

Example 1 (One Bond)

Assume one bond with a 0.7% probability to default in one year in all other cases it pays 105% in one year. What is the *VaR*?

[$\mathcal{A}$] The 1% VaR is -5% $\Rightarrow$ VaR spots **no** risk!

Example 2 (Two Independent Bonds)

Consider two identical bonds with the same parameters, but independently distributed. What is the *VaR* now?

[$\mathcal{A}$] The 1% VaR of the diversified portfolio is 47.5%!

The Cumulative Distribution Function

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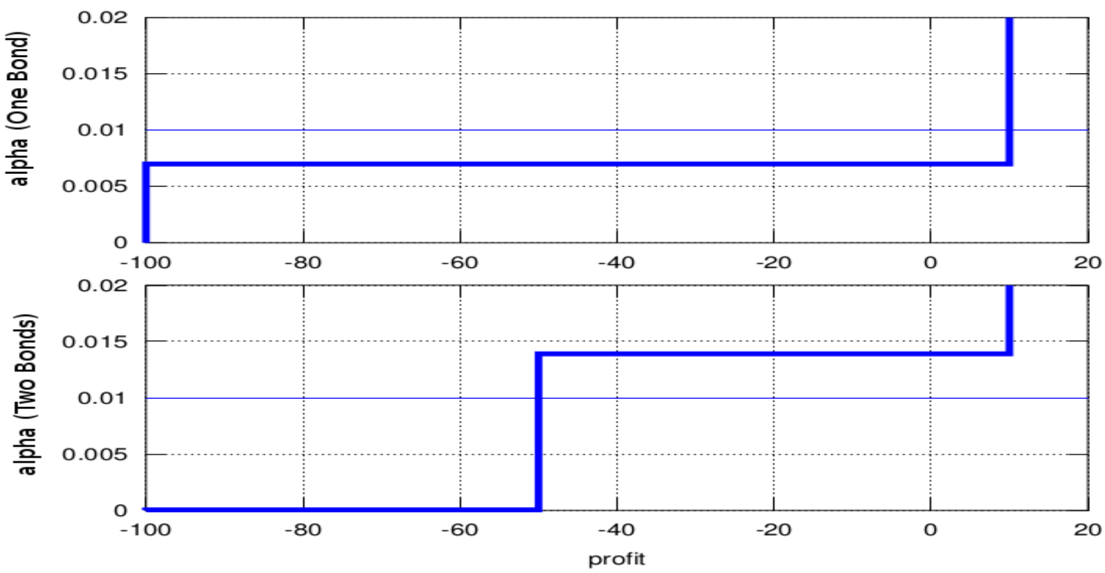

Figure 2: The cdf of $\mathcal{P}$ for one and two independent bonds.

Continuity in α

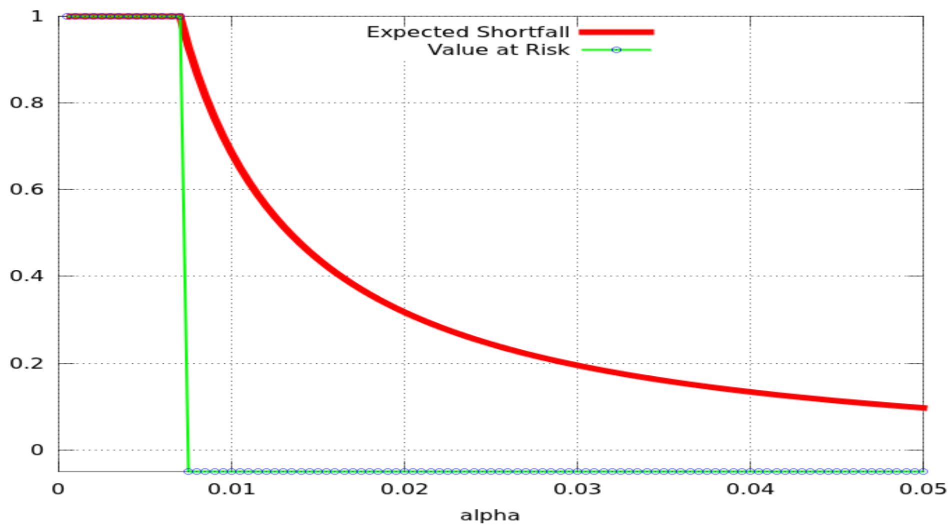

Figure 3: ES and VaR in function of α for one bond.

Case 1
Bonus Example

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Example 3 (The Evil Banker's First Dilemma)

Consider an Evil Banker who has to compose a portfolio for his private client. If there is at least one default in the portfolio, then the banker will loose that client.
How can our banker minimize his work and maximize his income?

[$\mathcal{A}$] The Evil Banker should minimize the probability that at least one bond defaults. This is:
$$P[\text{at least one default}] = 1 - \prod_{n=1}^{N} P[\text{one default}] = 1 - (0.7)^N.$$
The optimal value is hence $N = 1$.

Case 1
Bonus Example

Example 4 (The Evil Banker's Second Dilemma)

Consider an Evil Banker who hast to comply with Basel III, hence uses for assessing market risk *VaR*. Being Evil he does not care about the size of a bailout. So how does he minimize VaR?

[$\mathcal{A}$] One bond is optimal. However, *VaR* only informs that there is 1% chance that the loss will be higher than the VaR. The Evil Banker does not care, but the society should care about the size of an eventual bailout.

Case 2
More Bonds

Example 5 (N Independent Bonds)

Consider now an increasing number of independent bonds with the same parameters as in previous example.
Trace the risk surface.

Risk in Function of Diversification

Convecity (I)

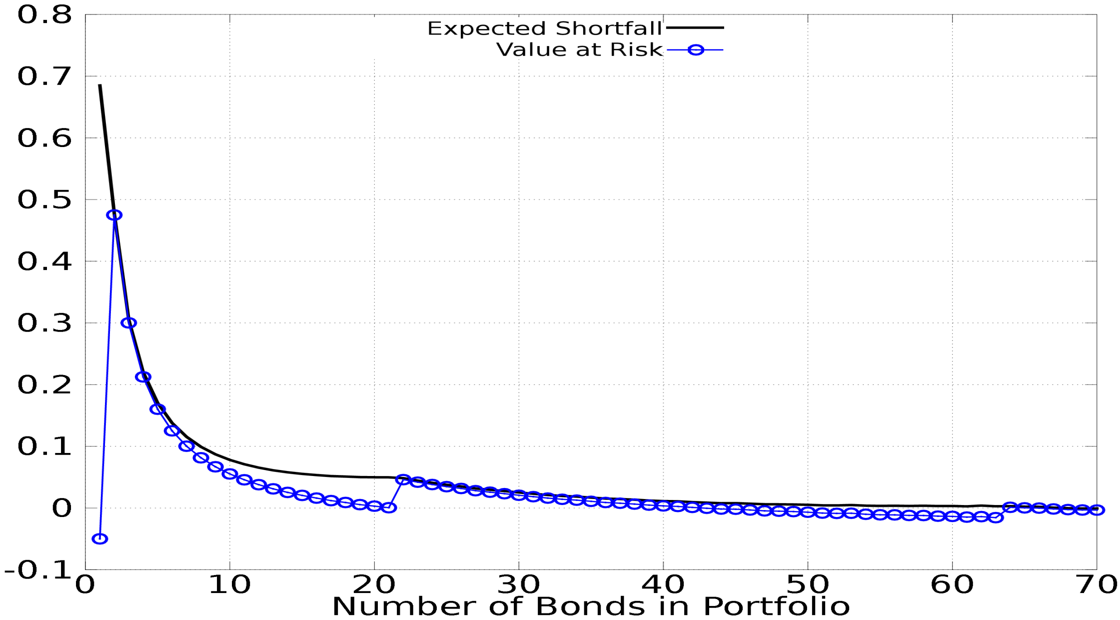

Figure 4: ES and VaR in function of number of bonds.

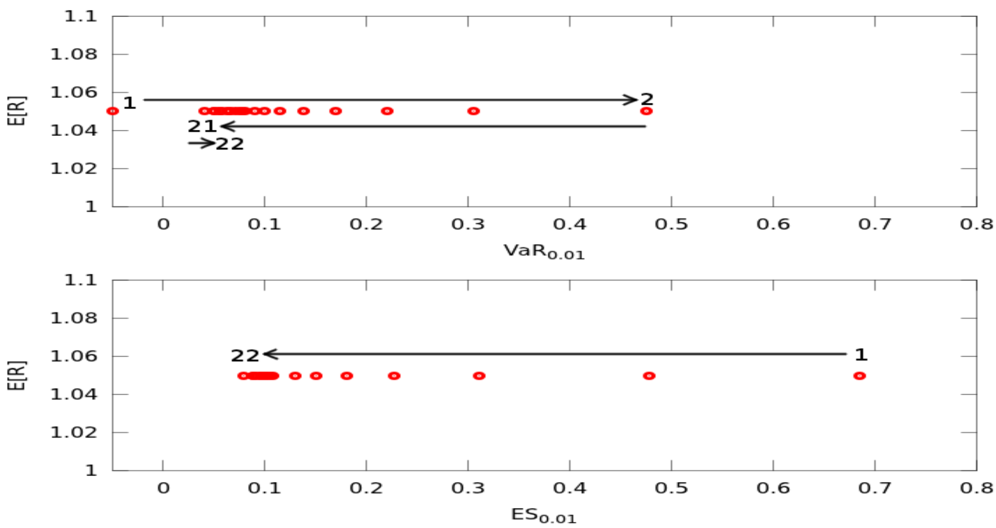

Figure 5: The result on the risk surface.

Case 3

Risk-Reward Optimization for Gaussian Returns

Example 6 (Three Gaussian Assets)

Consider three assets (or asset classes) that are all Gaussian (or at least elliptically) distributed and consider a risk-reward optimization

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Optimal Portfolio Composition

The Mechanics of a Risk-Reward Method

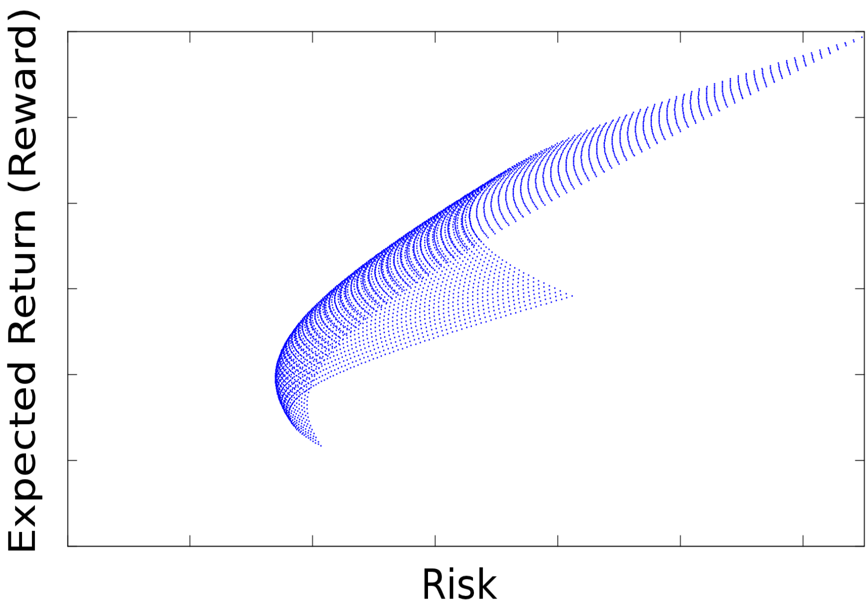

Figure 6: Portfolios in the risk/reward plane.

EXAMPLE 1
GAUSIAN EQUITIES, BONDS AND CASH—INFLATION ADJUSTED

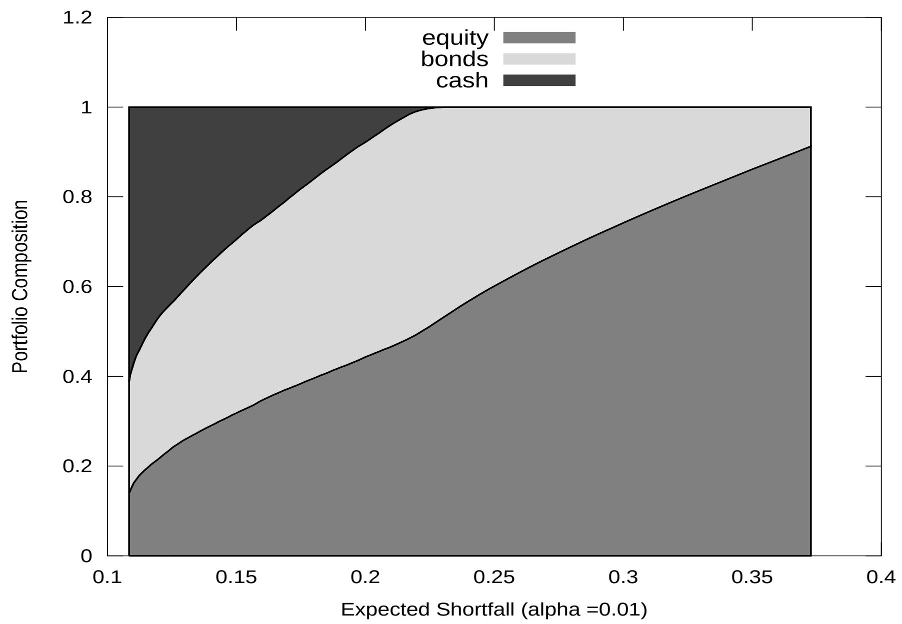

FIGURE 7: Recommended portfolios in function of ES.

Note that for Gaussian assets σ, *VaR* and *ES* lead to the same optimal portfolios.

CASE 4
RISK-REWARD OPTIMIZATION FOR NON-GAUSSIAN RETURNS

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EXAMPLE 7 (NON-GAUSSIAN ASSETS)

Consider three assets (or asset classes) that are all Gaussian distributed and consider a risk-reward optimization, but add a typical hedge fund and a typical capital guaranteed structure.

CASE 4: NON-GAUSSIAN ASSETS

THE PDFS

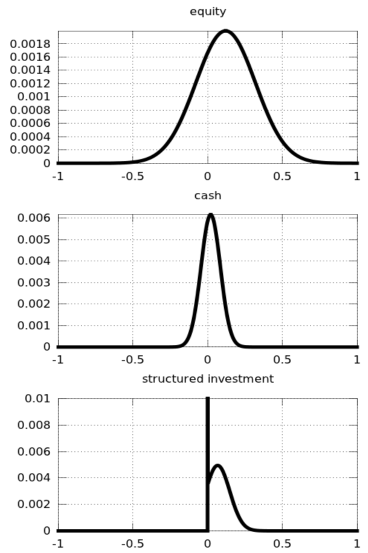

FIGURE 8: The pdfs in the example (the y-axis for the structured fund is truncated—this investment is a long call plus a deposit).

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CASE 4: NON-GAUSSIAN ASSETS

MEAN-ES AND MEAN-VAR OPTIMIZATION

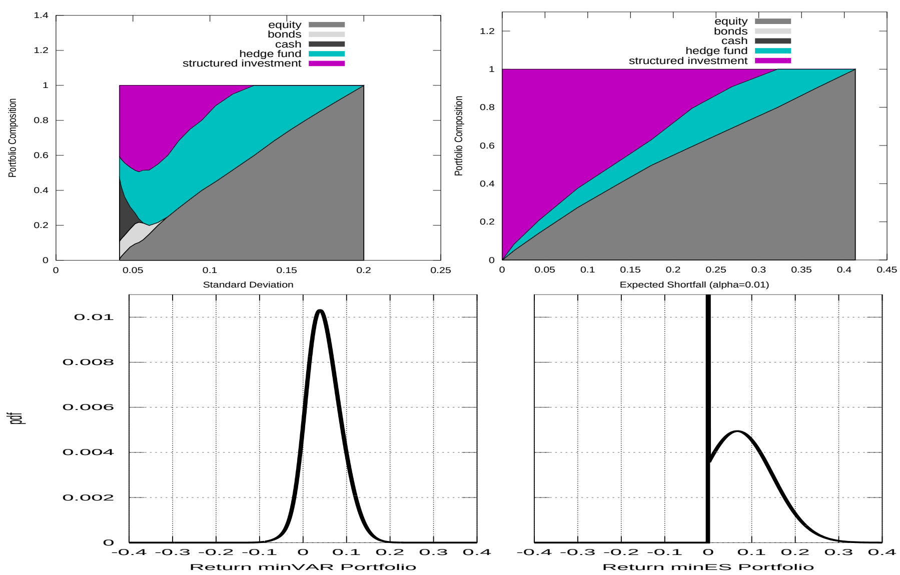

FIGURE 9: The min-VAR and min-ES portfolios compared.

Case 5 I

VaR as Risk Limit (UCITS IV)

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For UCITS that are not managed relative to a benchmark UCITS IV defines the "Absolute VaR" limit:

$$VaR_{UCITS} \leq 20\% NAV$$

Example 8 (Risky Bet Fund)

Consider a structured fund that will pay on one year time 105% of the initial investment (assume that it pays the capital back plus a coupon of 5% in one year), except if company X defaults in that year, then it pays 0%. We estimate the probability that company X defaults in one year to equal 0.7%.
The VaR_{UCITS} is -5%, so this is perfectly acceptable according to the General Guidelines of CESR/10-788.

Case 5 II

VaR as Risk Limit (UCITS IV)

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Example 9 (Better Diversified Fund)

Consider a structured fund that will pay on one year time 105% of the initial investment, if either company X or Y defaults then it pays 52.5% of the initial investment, and if both companies X and Y default then it pays zero. We estimate the default probability of both company X and Y to equal 0.7%, and their default possibility is independently distributed.
The VaR_{UCITS} is 47.5%, so this is not acceptable according to the General Guidelines of CESR/10-788.

Note: the same holds for the VaR limit in Basel II ICAAP.
Examples: Lehman Brothers, Dexia, ...

Case 6
A Risk Reward Indicator Based on Volatility (UICTS IV)

UCITS IV defines the "Risk Reward Indicator" as follows.

risk class	volatility equal or above	volatility less than
1	0%	0.5%
2	0.5%	2.0%
3	2.0%	5.0%
4	5.0%	10.0%
5	10.0%	15.0%
6	15.0%	25.0%
7	25.0%	$+\infty$

Table 1: The "risk classes" as defined by CESR in CESR/10-673, pg. 7, in the same document the risk classes are *also* referred to as "risk and reward indicator".

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Case 6
A Risk Reward Indicator Based on Volatility (UICTS IV)

Example 10 (Risk Classification)

Assume the assets from Example 1 plus one "risky bond" (this could also be a structured fund based on a digital option) that has a probability of 1% to loose 15% and a probability of 99% to gain 5%. Then consider the risk class as defined by CESR/10-673. The results are as in Table 2.

Case 6

A Risk Reward Indicator based on Volatility (UICTS IV)

portfolio	risk class	σ	$ES_{0.01}$
equity	6	0.2000	0.4123
bonds	5	0.1200	0.2660
hedge fund	5	0.1062	0.5482
structured investment	4	0.0671	0.0000
risky bond	2	0.0198	0.1500
mix 1/2 equity + 1/2 bonds	5	0.1173	0.2223

Table 2: The risk classes for Example 3. CESR/ESMA's method considers the hedge fund that has roughly a 2.5% probability of loosing about 50% of its value is in the same risk class as a bond fund. A structured fund that has no risk to lose something ends up in the fourth risk class, but the risky bond that has a 1% probability of loosing 15% is considered as very safe!

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Case 6

Bonus Example

Example 11 (The Evil Banker's Third Dilemma)

How to reduce the risk class of the “risky bond” structure?

[$\mathcal{A}$] The Evil Banker will reduce the maximal payoff of the structure and increase the management fee. This will reduce the volatility (but also the expected payoff). This trick would not work with a coherent risk measure.

Case 7

Incoherence between the VaR-limit and the VaR-risk-class

risk limit, based on VaR

$\Longleftrightarrow$

risk classification, based on standard deviation

Example 12

Consider a structured fund that offers a 1% probability to loose 21% and a 99% probability to gain 5%. Such fund would not be possible, because its 1% VaR_{UCITS} would be 21% (exceeding the limit and being classified as "too risky"). Its volatility is 2.5870%, that is only risk class 3, hence considered as safer than bonds—from our example, in the middle of the spectrum, and perfectly acceptable.

Section 4

The Limits of Coherent Risk Measures

EXAMPLE 13 (ILLIQUID ASSETS)

Imagine that you hold twice the average daily volume in stock X. Is it realistic to demand from a risk measure that it is positive homogeneous and hence that $\forall a > 0$ and $\forall X, aX \in \mathbb{V} : \rho(aX) = a\rho(X)$?

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EXAMPLE 14 (THIRSTY)

Imagine that you need to drink in order to cross the desert, but you know that one of your five bottles is poisoned (of course you don't know which one). What strategy do you take to minimize risk? Diversify or Russian Roulette?

The Limits of Coherent Risk Measures

Systemicity

Example 15 (Basel II with ES?)

Would it make sense to replace *VaR* in the capital requirements for banks by *ES*?

[$\mathcal{A}$] It would be a significant improvement, but would it also not work systemic? (i.e. act as a non-linear feedback system in case of disaster)

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The Limits of Coherent Risk Measures

Decreasing Utility of Money

Example 16 (Decreasing Marginal Utility)

Coherent risk measures do not seem to be congruent with decreasing marginal utility. Would it make sense to relax the homogeneity axiom when modelling preferences?

[$\mathcal{A}$] An individual might not care wether he or she defaults with an high or a very high amount. However the society that will have to cover for the fallout should care and demand homogeneity.

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EXAMPLE 17 (RISK AND REWARD INDICATOR?)

Could a coherent risk measure be a "risk and reward indicator"?

[$\mathcal{A}$] Stochastic Dominance of Second Order implies dominance of *ES* (Yamai and Yoshiba 2002). However for *ES* to imply stochastic dominance of the second order–and hence imply preference in utility theory–one would need an infinite number of *ES* calculations for all $\alpha \in [0, 1]$.

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Conclusions

1. The use of an incoherent risk measure will inevitably lead to counter-intuitive an dangerous results.
2. It seems to make sense to make rough assumptions about the left tail of the return distribution rather than to ignore it altogether.
3. Coherence does matter and its importance cannot be underestimated.

Back-matter

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SECTION 8

NOMENCLATURE

Nomenclature I

α	a probability $\in [0,1]$, page 7
$\mathbb{V}$	the set of real-valued stochastic variables, page 7
$\mathcal{L}$	loss (expressed in monetary terms), $\mathcal{L} = -\mathcal{P}$, page 7
$\mathcal{P}$	profit (expressed in monetary terms), $\mathcal{P} = -\mathcal{L}$, page 7
ρ	a risk measure, $\rho : \mathbb{V} \mapsto \mathbb{R}$, page 13
$E[X]$	the expected value of a stochastic variable X: $E[X] = \int f_X(x)x\,\mathrm{d}x$, page 7
$ES_\alpha(\mathcal{P})$	Expected Shortfall = the average of the $\alpha 100\%$ worst outcomes of $\mathcal{P}$; aka CVaR, Tail-VaR, etc., page 11
$f_X(.)$	the probability density function of the stochastic variable X, page 7
$Q_X(\alpha)$	the quantile function of the stochastic variable X, page 9
S	semi-variance, $S := E[\min(0, R-c)^2]$, page 5
$VAR(X)$	Variance: $VAR(X) = E[X^2] - E[X]^2 = \sigma^2$, page 7
$VaR_\alpha(\mathcal{P})$	Value at Risk, page 9
pdf	probability density function, page 29